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Hull number: P_5-free graphs and reduction rules*

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Thème : Algorithmique, programmation, logiciels et architectures
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Abstract: In this paper, we study the (geodesic) hull number of graphs. For any two vertices $u, v \in V$ of a connected undirected graph $G = (V, E)$, the closed interval $I[u, v]$ of u and v is the set of vertices that belong to some shortest (u, v)-path. For any $S \subseteq V$, let $I[S] = \bigcup_{u,v \in S} I[u, v]$. A subset $S \subseteq V$ is (geodesically) convex if $I[S] = S$. Given a subset $S \subseteq V$, the convex hull $I_h[S]$ of S is the smallest convex set that contains S. We say that S is a hull set of G if $I_h[S] = V$. The size of a minimum hull set of G is the hull number of G, denoted by $hn(G)$.

First, we show a polynomial-time algorithm to compute the hull number of any P_5-free triangle-free graph. Then, we present four reduction rules based on vertices with the same neighborhood. We use these reduction rules to propose a fixed parameter tractable algorithm to compute the hull number of any graph G, where the parameter can be the size of a vertex cover of G or, more generally, its neighborhood diversity, and we also use these reductions to characterize the hull number of the lexicographic product of any two graphs.

Key-words: Graph Convexity, Hull Number, Geodesic Convexity, P_5-free Graphs, Lexicographic Product, Parameterized Complexity, Neighborhood Diversity.

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Nombre enveloppe : graphes sans P_5 et règles de réduction

Résumé : Dans cet article, nous étudions le *nombre enveloppe* (géodésique) des graphes. Pour deux sommets u et $v \in V$ d'un graphe connexe non orienté $G = (V, E)$, l'intervalle fermé $I[u, v]$ de u et v est l'ensemble des sommets qui appartiennent à une plus courte chaîne reliant u et v. Pour tout $S \subseteq V$, on note $I[S] = \bigcup_{u,v \in S} I[u, v]$. Un sous-ensemble $S \subseteq V$ est (géodésiquement) convexe si $I[S] = S$. Étant donné un sous-ensemble $S \subseteq V$, l'enveloppe convexe $I_h[S]$ de S est le plus petit ensemble convexe qui contient S. On dit que S est un *ensemble enveloppe* de G si $I_h[S] = V$. La taille d'un ensemble enveloppe minimum de G est le nombre enveloppe de G, noté $hn(G)$.

Tout d'abord, nous donnons un algorithme polynomial pour calculer le nombre enveloppe d'un graphe sans P_5 et sans triangle. Ensuite, nous présentons quatre règles de réductions basées sur des sommets ayant même voisinage. Nous utilisons ces règles de réduction pour proposer un algorithme FPT pour calculer le nombre enveloppe de n'importe quel graphe G, ou le paramètre peut être la taille d'un transversal de G ou, plus généralement sa diversité de voisinage ; nous utilisons également ces règles pour caractériser le nombre enveloppe du produit lexicographique de deux graphes.

Mots-clés : Convexité dans les graphes, Nombre enveloppe, Convexité géodésique, Graphes sans P_5, Produit lexicographique, Complexité paramétrée, Diversité de voisinage.

1 Introduction

All graphs in this work are undirected, simple and loop-less. Given a connected graph $G = (V, E)$, the closed interval $I[u, v]$ of any two vertices $u, v \in V$ is the set of vertices that belong to some u-v geodesic of G, i.e. some shortest (u, v)-path. For any $S \subseteq V$, let $I[S] = \bigcup_{u,v \in S} I[u, v]$. A subset $S \subseteq V$ is *(geodesically) convex* if $I[S] = S$. Given a subset $S \subseteq V$, the *convex hull* $I_h[S]$ of S is the smallest convex set that contains S. We say that a vertex v is *generated* by a set of vertices S if $v \in I_h[S]$. We say that S is a *hull set* of G if $I_h[S] = V$. The size of a minimum hull set of G is the *hull number* of G, denoted by $hn(G)$ [9].

It is known that computing $hn(G)$ is an NP-hard problem for bipartite graphs [3]. Several bounds on the hull number of triangle-free graphs are presented in [8]. In [7], the authors show, among other results, that the hull number of any P_4-free graph, i.e. any graph without induced path with four vertices, can be computed in polynomial time. In Section 3, we show a linear-time algorithm to compute the hull number of any P_5-free triangle-free graph.

In Section 4, we show four reduction rules to obtain, from a graph G, another graph G^* that has one vertex less than G and which satisfies either $hn(G) = hn(G^*)$ or $hn(G) = hn(G^*) + 1$, according to the used rule. We then first use these rules to obtain a fixed parameter tractable (FPT) algorithm, where the parameter is the neighborhood diversity of the input graph. For definitions on Parameterized Complexity we refer to [10]. Given a graph G and vertices $u, v \in V(G)$, we say that u and v are of the *same type* if $N(v) \setminus \{u\} = N(u) \setminus \{v\}$. The *neighborhood diversity* of a graph is k, if its vertex set can be partitioned into k sets $S_1, \ldots, S_k$, such that any pair of vertices $u, v \in S_i$ are of the same type. This parameter was proposed by Lampis [12], motivated by the fact that a graph of bounded vertex cover also has bounded neighborhood diversity, and therefore the later parameter can be used to obtain more general results. To see that a graph of bounded vertex cover has bounded neighborhood diversity, let G be a graph that has a vertex cover $S \subseteq V(G)$ of size k, and let $I = V(G) \setminus S$. Since S is a vertex cover, observe that I is an independent set. Therefore, vertices in I can be partitioned in at most 2^k sets (one for each possible subset of S), where each of these sets contains vertices of the same type, i.e. vertices having the same neighborhood in S. Moreover, the vertices in S may be partitioned in k sets of singletons, what gives a partition of the vertices of the graph into $k + 2^k$ sets of vertices having the same type. Then, the neighborhood diversity of the graph is at most $k + 2^k$. Many problems have been show to be FPT when the parameter is the neighborhood diversity [11].

Finally, we use these rules to characterize the hull number of the lexicographic product of any two graphs. Given two graphs G and H, the *lexicographic product* $G \circ H$ is the graph whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and such that two vertices (g_1, h_1) and (g_2, h_2) are adjacent if, and only if, either $g_1g_2 \in E(G)$ or we have that both $g_1 = g_2$ and $h_1h_2 \in E(G)$.

It is known in the literature a characterization of the (geodesic) convex sets in the lexicographic product of two graphs [1] and a study of the pre-hull number for this product [13]. There are also some results concerning the hull number of the Cartesian and strong products of graphs [6, 5].

2 Preliminaries

Let us recall some definitions and lemmas that we use in the sequel.

We denote by $N_G(v)$ (or simply $N(v)$) the neighborhood of a vertex. A vertex v is *simplicial* (resp. *universal*) if $N(v)$ is a clique (resp. is equal to $V(G)\setminus\{v\}$). Let $d_G(u,v)$ denote the *distance* between u and v, i.e. the length of a shortest (u,v)-path. A subgraph $H \subseteq G$ is *isometric* if, for each $u, v \in V(H)$, $d_H(u,v) = d_G(u,v)$. A P_k (resp. C_k) in a graph G denotes an induced path (resp. cycle) on k vertices. Given a graph H, we say that a graph G is H-free if G does not contain H as an induced subgraph. Moreover, we consider that all the graph in this work are connected. Indeed, if a graph G is not connected, its hull number can be computed by the sum of the hull numbers of its connected components, as observed by Dourado et al. [7].

Lemma 1. *[9] For any hull set S of a graph G, S contains all simplicial vertices of G.*

Lemma 2. *[7] Let G be a graph which is not complete. No hull set of G with cardinality $hn(G)$ contains a universal vertex.*

Lemma 3. *[7] Let G be a graph, H be an isometric subgraph of G and S be any hull set of H. Then, the convex hull of S in G contains $V(H)$.*

Lemma 4. *[7] Let G be a graph and S a proper and non-empty subset of $V(G)$. If $V(G)\setminus S$ is convex, then every hull set of G contains at least one vertex of S.*

3 Hull number of P_5-free triangle-free graphs

In this section, we present a linear-time algorithm to compute $hn(G)$, for any P_5-free triangle-free graph G. To prove the correctness of this algorithm, we need to recall some definitions and previous results:

Definition 1. *Given a graph $G = (V,E)$, we say that $S \subseteq V$ is a* dominating set *if every vertex $v \in V \setminus S$ has a neighbor in S.*

It is well known that:

Theorem 1. *[4] G is P_5-free if, and only if, for every induced subgraph $H \subseteq G$ either $V(H)$ contains a dominating C_5 or a dominating clique.*

As a consequence, we have that:

Corollary 1. *If G is a connected P_5-free bipartite graph, then there exists a dominating edge in G.*

Theorem 2. *The hull number of a P_5-free bipartite graph $G = (A \cup B, E)$ can be computed in linear time.*

Proof. By Corollary 1, G has at least one dominating edge. Observe that the dominating edges of a bipartite graph can be found in linear time by computing the degree of each vertex and then considering the sum of the degrees of the endpoints of each edge. For a dominating edge, this sum is equal to the number of vertices.

- Consider first the case in which G has at least two dominating edges. Let $uv, xy \in E(G)$ be such dominating edges. Consider that $u, x \in A$ and $v, y \in B$.

  If $x \neq u$ and $v \neq y$, then we claim that $\{u, x\}$ is a minimum hull set of G. Indeed, since u and x are not adjacent and every vertex in B is a common neighbor of u and x, and then $\{u, x\}$ generate all the vertices in B, particularly v and y. Similarly, all the vertices of A are in a shortest (v, y)-path. Thus, $I_h(\{u, x\}) = V(G)$.

  Assume now, w.l.o.g., that $u \neq x$ and $v = y$. Again, $B \subseteq I_h(\{u, x\})$. Observe that, if there are simplicial vertices in $V(G)$, they must all belong to A, since u and x are not neighbors, but they are adjacent to all vertices in B. In case $|B| = 1$, then all vertices in A are simplicial vertices, and therefore A is the minimum hull set of G. Then, consider now that $|B| \geq 2$.

  In case there is no simplicial vertex in A, $\{u, x\}$ is a minimum hull set, since $B \subseteq I_h(\{u, x\})$ and every vertex in A has at least two neighbors in B. In case there are simplicial vertices in A, we claim that $S \cup \{b\}$ is a minimum hull set of G, where $S \subset A$ is the set of simplicial vertices of G and b is a vertex in B distinct from v. Indeed, by Lemma 1, we know that S must be part of any hull set of G and observe that $I_h(S) = S \cup \{v\}$ (the only neighbor of each simplicial vertex is exactly v). Consequently, since $|B| \geq 2$, at least one more vertex must be chosen to be part of a minimum hull set of G. We claim that if we choose $b \in B$ such that $b \neq v$, then $S \cup \{b\}$ is a minimum hull set of G. Indeed, let $s \in S$. Since $sb \notin E$ and xv, uv are dominating edges, x, u and v are generated by $\{s, b\}$. But then, as $B \subseteq I_h(\{u, x\})$, B is generated. Finally, every vertex in A is either simplicial, in case it belongs to S, or is adjacent to two vertices in B and therefore is generated by its neighbors.

- Consider now that G has only one dominating edge uv and that, w.l.o.g., $u \in A$ and $v \in B$. Let $H = G[V \setminus \{u, v\}]$. We may assume H is not the empty graph, for otherwise G is trivial. Let $C_1, \ldots, C_k$ be the connected components of H. We claim that $V \setminus C_i$ is a convex set of G, for every $i \in \{1, \ldots, k\}$.

  Since C_i is a connected component in H, the only vertices in $V \setminus C_i$ that may be adjacent to a vertex in C_i are u and v. Suppose a shortest (s, t)-path P such that $s, t \in V \setminus C_i$ and containing at least one vertex of C_i. It would pass through u and v. But there is an edge between u and v, so there is a contradiction because P would not be a shortest path.

  Consequently, by Lemma 4, for each connected component C_i of H at least one vertex of C_i must be chosen to be part of a minimum hull set of G (observe that simplicial vertices are the particular case in which $|C_i| = 1$).

  If $k = 1$, observe that G is not a complete bipartite graph, as we are assuming there is exactly one dominating edge. Let $w \in A$ and $z \in B$ be two non-adjacent vertices of C_1. In this case, we claim that $\{w, z\}$ is a minimum hull set of G. By contradiction, suppose that there exists a vertex $p \notin I_h(\{w, z\})$. First observe that u and v belong to $I_h(\{w, z\})$. Then, w.l.o.g., we may assume that p has a neighbor q in $I_h(\{w, z\})$ which is not in $\{u, v\}$, since C_1 is a connected component in H. However, since uv is a dominating edge, either qpu or qpv is a shortest path between two vertices of $I_h\{w, z\}$ and p should belong to $I_h(\{w, z\})$, a contradiction.

Now, suppose that $k > 1$. Let $W = \{w_1, \ldots, w_k\} \subseteq V(G)$ be such that $W \cap A \neq \emptyset$, $W \cap B \neq \emptyset$ and $w_i \in C_i$, for every $i \in \{1, \ldots, k\}$. We claim that W is a minimum hull set of G. By Lemma 4, all these vertices are required, so it suffices to show that $I_h(W) = V(G)$. Observe that u and v belong to $I_h(W)$, since $W \cap A \neq \emptyset$ and $W \cap B \neq \emptyset$. Then, by contradiction, suppose that there exists a vertex $p \notin I_h(W)$ and let C_p be its connected component in H. Again, we may assume that p has a neighbor q in $I_h(\{w, z\})$ which belongs to C_p, since C_p is a connected component and $C_p \cap W \neq \emptyset$. However, since uv is a dominating edge, either qpu or qpv is a shortest path in G and p should belong to $I_h(\{w, z\})$, a contradiction.

Finally, observe that all these cases can be checked in linear time and thus $hn(G)$ can be computed in linear time. □

For the next result, recall that the complexity of finding the convex hull of a set of vertices $S \subseteq V(G)$ of a graph G is $\mathcal{O}(|S||E(G)|)$, as described in [7]. We can relax the constraint of G being bipartite to obtain the following:

Corollary 2. *If G is a P_5-free triangle-free graph, then $hn(G)$ can be computed in polynomial time.*

Proof. By Theorem 1, G either has a dominating C_5 or a dominating clique of size at most two, since it is triangle-free.

In case it has a dominating $C_5 = v_1, \ldots, v_5$, we claim that $\{v_1, v_3, v_5\}$ is a hull set of G. To prove this fact, first observe that $I_h(\{v_1, v_3, v_5\}) \supseteq V(C_5)$. Moreover, since G is connected, and it has no induced P_5 and no triangle, we conclude that any vertex $w \in V(G) \setminus V(C_5)$ has two non-adjacent neighbors in C_5, and so $w \in I_h(\{v_1, v_3, v_5\})$. Thus, if G has a dominating C_5, we can test if there is a minimum hull set of size two in $\mathcal{O}(|V(G)|^2|E(G)|)$. Otherwise, we have that $hn(G) = 3$ and $\{v_1, v_3, v_5\}$ is a minimum hull set of G.

If G has a dominating clique of size one, then G must be a star since it is triangle-free. Thus, $hn(G) = |V(G)| - 1$.

Finally, if G has a dominating edge uv, we claim that G is bipartite. Since G is triangle-free and uv is a dominating edge, we have that $N(u)$ and $N(v)$ are stable sets and that $N(u) \cap N(v) = \emptyset$. Thus, G is bipartite and, by Theorem 2, we can compute its hull number in linear time. □

4 Neighborhood Diversity and Lexicographic Product

In this section, we present four reduction rules to compute the hull number of a graph. We need to introduce some definitions.

Given a set S, let $I^0[S] = S$ and $I^k[S] = I[I^{k-1}[S]]$, for $k > 0$. We say that v is generated by S *at step* $t \geq 1$, if $v \in I^t[S]$ and $v \notin I^{t-1}[S]$. Observe that the convex hull $I_h(S)$ of a given set of vertices S is equal to $I^{|V(G)|}[S]$.

Given a graph G, we say that two vertices v and v' are *twins* if $N(v) \setminus \{v'\} = N(v') \setminus \{v\}$. If v and v' are adjacent, we call them *true* twins, otherwise we say that they are *false* twins.

Let G be a graph and v and v' be two of its vertices. The *identification* of v' into v is the operation that produces a graph G' such that $V(G') = V(G) \setminus \{v'\}$ and $E(G') = (E(G) \setminus \{v'w \mid w \in N_G(v')\}) \cup \{vw \mid v'w \in E(G) \text{ and } w \neq v\}$.

Lemma 5. *Let G be a graph and v and v' be non-simplicial and twin vertices. Let G' be obtained from G by the identification of v' into v. Then, $hn(G) = hn(G')$.*

Proof. Let u and w be two non-adjacent neighbors of v and thus also of v' in G. In order to show that $hn(G) \leq hn(G')$, let S be a minimum hull set of G'. Since G' is an isometric subgraph of G, $V(G) \setminus \{v'\} \subseteq I_h(S)$ by Lemma 3. Moreover, $\{v'\} \subseteq I_G[u, w]$, hence S is a hull set of G.

To prove that $hn(G) \geq hn(G')$, let S be a minimum hull set of G. We may assume that S does not contain both v and v', because if there exists a minimum hull set containing both of them, then we can replace v and v' by u and w obtaining a hull set of same size, since $v, v' \in I_G[u, w]$.

Suppose first that $v, v' \notin S$. Let $\{x, y\} \neq \{v, v'\}$ and let P be a shortest (x, y)-path. Observe that P cannot contain both v and v'. In case v' (resp. v) is contained in P, then one can replace it by v (resp. v') and obtain another shortest path, as v and v' have the same neighborhood. In particular, this implies that the minimum k such that $v' \in I^k_G[S]$ is equal to the minimum k' such that $v \in I^{k'}_{G'}[S]$, and therefore for $i < k$, $I^i_{G'}[S] = I^i_G[S]$. It also implies that $I_G[v', w] \setminus \{v'\} = I_{G'}[v, w] \setminus \{v\}$, $w \notin \{v, v'\}$, and therefore for $i \geq k$ we have that $I^i_{G'}[S] = I^i_G[S] \setminus \{v'\}$. As a consequence, S is a hull set of G'.

Finally, suppose that either v or v' is in S. We may assume w.l.o.g. that $v \in S$. Then we can use the same argument as in the last paragraph to show that for every $1 \leq i \leq n$ its true that $I^i_{G'}[S] = I^i_G[S] \setminus \{v'\}$ and then again we have that S is a hull set of G'. □

Lemma 6. *Let G be a graph and v, v', v'' be simplicial and pairwise false twin vertices. Let G' be obtained from G by the identification of v'' into v. Then, $hn(G) = hn(G') + 1$.*

Proof. In order to show that $hn(G) \leq hn(G')+1$, observe that G' is an isometric subgraph of G and that v'' is simplicial. Consequently, any hull set S of G' is such that $I_h(S) = V(G) \setminus \{v''\}$, hence $S \cup \{v''\}$ is a hull set of G, by Lemmas 1 and 3.

To show that $hn(G) \geq hn(G')+1$. Let S be a hull set for G and $S' = S \setminus \{v''\}$. Since v, v' and v'' are simplicial, we know that $\{v, v', v''\} \subseteq S$. Any shortest (v'', u)-path, with $u \in V \setminus \{v', v''\}$ is still a shortest path if v'' is replaced by v', so $I[v'', u] = I[v', u]$. In the case of the shortest (v'', v')-path, replacing v'' by v is still a shortest path and $I[v'', v'] = I[v, v']$. Therefore $I_h(S') = I_h(S) \setminus \{v''\}$ and then S' is a hull set of G'. □

Observe that we cannot simplify the statement of Lemma 6 to consider any pair of simplicial false twin vertices instead of triples. As an example, consider the graph obtained by removing an edge uv from a complete graph with more than 3 vertices.

Lemma 7. *Let G be a graph and v, v' be simplicial and true twin vertices. Let G' be obtained from G by the identification of v' into v. Then, $hn(G) = hn(G')+1$.*

Proof. In order to show that $hn(G) \leq hn(G')+1$, observe that G' is an isometric subgraph of G and that v' is simplicial. Let S be a hull set of G'. Then $S \cup \{v'\}$ is a hull set of G, by Lemmas 1 and 3.

Now, we show that $hn(G) \geq hn(G') + 1$. Let S be a hull set of G. Since v and v' are simplicial, by Lemma 1 we know that $v, v' \in S$. Observe that, for every $w \in V(G')$, we have $I_G[v', w] \setminus \{v'\} \subseteq I_{G'}[v, w]$. Thus, $S \setminus \{v'\}$ is a hull set of G' and the result follows. $\square$

According to Lampis [12], for a given graph G and vertices $u, v \in V(G)$, u and v are of the *same type* if $N(v) \setminus \{u\} = N(u) \setminus \{v\}$. This is exactly the same definition of *twin* vertices. Recall that the *neighborhood diversity* of a graph is k, if its vertex set can be partitioned into k sets $S_1, \ldots, S_k$, such that any pair of vertices $u, v \in S_i$ are of the same type. Now, we use this partition to obtain the following result:

Theorem 3. *Let G be a graph whose neighborhood diversity is at most k. Then, there exists an FPT algorithm to compute $hn(G)$ in $\mathcal{O}(4^k poly(|V(G)|))$-time.*

Proof. Lampis proved that a neighborhood partition of G can be found in $\mathcal{O}(poly(|V(G)|))$-time [12]. Observe that each part is either an independent set of false twin vertices or a clique of true twin vertices. We now use Lemmas 5, 6 and 7 to reduce each of these parts to at most two vertices.

First, in case there are parts of size greater than one consisting of non-simplicial vertices, we reduce these parts to a single vertex by the identification of its vertices. This procedure generates a graph G' whose hull number is equal to $hn(G)$, by Lemma 5.

Observe that if a vertex is simplicial, then its part is composed of simplicial vertices. In the sequence, we reduce each part of size greater than two containing only independent simplicial false twins to two vertices, by applying Lemma 6. If c identifications are done in this procedure, then the hull number of the graph G'' obtained after this procedure is $hn(G'') = hn(G') - c = hn(G) - c$.

Then, we reduce all the parts composed of pairwise adjacent simplicial true twins to one vertex, by applying Lemma 7. In the end of this procedure, we obtain a graph G''' such that $hn(G''') = hn(G'') - c' = hn(G) - c - c'$, where c' is the number of identifications that were made in this last procedure.

Observe that G''' has at most $2k$ vertices, since the neighborhood partition is of size at most k and each part is reduced to at most two vertices. Finally, we can enumerate all the subsets of $V(G''')$ (there are at most 2^{2k} of them) and test for each of these sets whether it is a hull set. Hence, we obtain $hn(G''')$ and therefore $hn(G)$.

Recall that this proof provides a kernelization algorithm and G''' is a kernel of linear size. $\square$

As pointed before, a graph of bounded vertex cover size has also bounded neighborhood diversity, therefore the previous result also holds for this parameter.

Now, we use Lemma 5 and Lemma 7 to determine the lexicographic product of two graphs. Recall that the lexicographic product of two graphs G and H is the graph whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and such that two vertices (g_1, h_1) and (g_2, h_2) are adjacent if, and only if, either $g_1 g_2 \in E(G)$ or we have both $g_1 = g_2$ and $h_1 h_2 \in E(G)$. For a vertex $g \in V(G)$, let its *H-layer* in $G \circ H$ be the set $H(g) = \{(g, h) \in V(G \circ H) \mid h \in V(H)\}$. Let $S(G)$ denote the set of simplicial vertices of G.

Observe that if G has a single vertex, then $hn(G \circ H) = hn(H)$. Else, we have that:

Theorem 4. *Let G be a connected graph, such that $|V(G)| \geq 2$, and let H be an arbitrary graph. Thus,*

$$hn(G \circ H) = \begin{cases} 2, & \text{if } H \text{ is not complete;} \\ (|V(H)| - 1)|S(G)| + hn(G), & \text{otherwise.} \end{cases}$$

Proof. If H is not complete, since G is connected and it has at least two vertices, any two non-adjacent vertices in the same H-layer suffice to generate all the vertices of $G \circ H$.

We consider now that H is a complete graph on k vertices. First, observe that all the vertices in the same H-layer are all simplicial vertices or they are all non-simplicial vertices. Moreover, a vertex is simplicial in G if, and only if, its corresponding H-layer in $G \circ H$ is composed of simplicial vertices.

First, we obtain from $G \circ H$ a graph F by reducing each H-layer composed of non-simplicial vertices to a single vertex. By Lemma 5, $hn(G \circ H) = hn(F)$. Then, we apply Lemma 7 to reduce each H-layer of simplicial vertices to a single vertex obtaining a graph F'. Observe that we have $|V(H)||S(G)|$ simplicial vertices in $G \circ H$ and, thus, $(|V(H)| - 1)|S(G)|$ identifications are done in this procedure. Finally, since all the H-layers were reduced to a single vertex, observe that $F' \cong G$ and we have that $hn(G \circ H) = hn(F) = hn(F') + (|V(H)| - 1)|S(G)| = hn(G) + (|V(H)| - 1)|S(G)|$.

□

5 Conclusions

In this work, we first presented a linear time algorithm to compute the hull number of any P_5-free triangle-free graph. Although, the computational complexity of determining the hull number of a P_5-free graph and also of a triangle-free graph is still unknown. More generally, we propose the following open question:

Question 1. *For a fixed k, what is the computational complexity of determining $hn(G)$, for a P_k-free graph G?*

In the second part of this paper, we introduced four reduction rules that we use to present an FPT algorithm to compute the hull number of any graph, where the parameter is its neighborhood diversity, and a characterization of the lexicographic product of any two graphs. It is already known in the literature another FPT algorithm to compute the hull number of any graph, where the parameter is the number of its induced P_4's [2]. To the best our knowledge, the following is also open:

Question 2. *Given a graph G, is there an FPT algorithm to determine whether $hn(G) \leq k$, for a fixed k?*

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